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A Problem of Zarankiewicz

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Zarankiewicz, in problem P 101, *Colloq. Math.*, **2** (1951), p. 301, and others have posed the following problem: Determine the least positive integer $k_{\alpha,\beta}(m, n)$ so that if a 0,1-matrix of size m by n contains $k_{\alpha,\beta}(m, n)$ ones then it must have an α by β submatrix consisting entirely of ones. This paper improves upon previously known upper bounds for $k_{\alpha,\beta}(m, n)$ by proving that $k_{\alpha,\beta}(m, n) \leqslant 1 + ((\beta - 1)/\binom{p}{\alpha - 1})\binom{m}{\alpha} + ((p + 1)(\alpha - 1)/\alpha)n$ for each integer p greater than or equal to $\alpha - 1$. Each of these inequalities is better than the others for a specific range of values of n. Equality is shown to hold infinitely often for each value of p. Finally some applications of this result are made to arrangements of lines in the projective plane.

1. INTRODUCTION: DEFINITION OF PROBLEM AND SOME RECENT RESULTS

Zarankiewicz [10] and others have posed the following problem: Determine the least positive integer $k_{\alpha,\beta}(m, n)$ such that if a 0, 1-matrix of size m by n contains $k_{\alpha,\beta}(m, n)$ ones then it must have a submatrix of size α by β consisting entirely of ones. We restate the problem by asking for the largest positive integer $M_{\alpha,\beta}(m, n)$ so that there exists a 0, 1-matrix of size m by n with $M_{\alpha,\beta}(m, n)$ ones and no submatrix of size α by β consisting entirely of ones. Clearly, $M_{\alpha,\beta}(m, n) + 1 = k_{\alpha,\beta}(m, n)$.

First we state some recent results on the problem of Zarankiewicz. In Section 2 we prove the main result. We discuss equality in Section 3. Finally in Section 4 we show some connections with arrangements of lines in the projective plane.

Many results have already been established relating to the problem of Zarankiewicz. Kövari, Sös, and Turán [7] proved:

$$M_{2,2}(q^2 + q, q^2) = q^2(q + 1), \tag{1}$$

for q a prime number.

Reiman [8] showed

$$M_{2,2}(m, n) \leqslant \tfrac{1}{2}(m + (m^2 + 4mn(n - 1))^{1/2}), \tag{2}$$

with equality in infinitely many cases, e.g., in (1) and in

$$M_{2,2}(q^2+q+1, q^2+q+1) = q^3+2q^2+2q+1, \tag{3}$$

for q a prime power.

Hyltén-Cavallius [6] observed that (2) can be generalized to

$$M_{2,j}(m, n) = \tfrac{1}{2}((j-1)\,nm(m-1)+\tfrac{1}{4}n^2)^{1/2}. \tag{4}$$

Theorem 1 improves (4), hence must give equality in (1) and (3) (see Fig. 1).

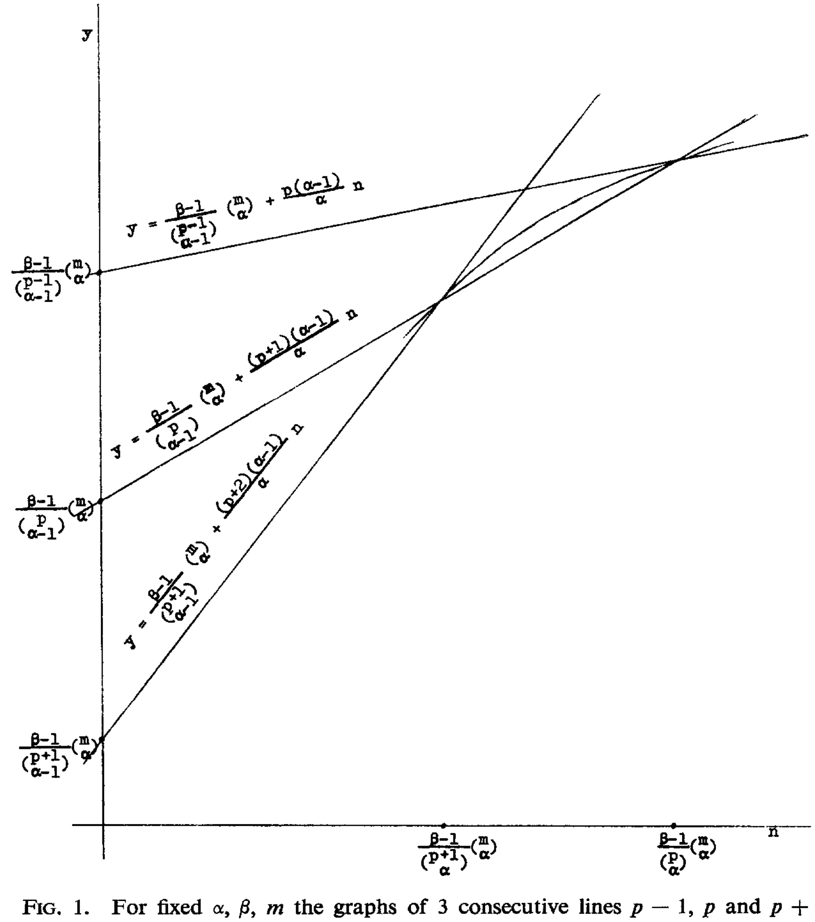

FIG. 1. For fixed α, β, m the graphs of 3 consecutive lines $p-1$, p and $p+1$ indicating the range of superiority for each p. The curved line is Eq. (2).

Čulík [3] has shown that for $n \geqslant (\beta - 1)\binom{m}{\alpha}$

$$M_{\alpha,\beta}(m, n) = (\alpha - 1)n + (\beta - 1)\binom{m}{\alpha}. \tag{5}$$

R. K. Guy [4] has proved that for $\alpha = 2, 3$

$$M_{\alpha,\beta}(m, n) = ((\alpha^2 - 1)/\alpha)n + ((\beta - 1)/\alpha)\binom{m}{\alpha}, \tag{6}$$

whenever $l(m, \alpha, \beta) \leqslant n \leqslant (\beta - 1)\binom{m}{\alpha}$, where $l(m, \alpha, \beta)$ has the approximate value $(\beta - 1)/(\alpha + 1)\binom{m}{\alpha}$. He has also asserted Eq. (6) for all α and β. Our Theorem 2 establishes (6) for general α and β.

2. The Main Result

THEOREM 1. *If $M_{\alpha,\beta}(m, n)$ is the largest positive integer such that there exists a* 0, 1-*matrix of size m by n with $M_{\alpha,\beta}(m, n)$ ones and no submatrix of size α by β consisting entirely of ones then*

$$M_{\alpha,\beta}(m, n) \leqslant \frac{\beta - 1}{\binom{p}{\alpha - 1}}\binom{m}{\alpha} + \frac{(p + 1)(\alpha - 1)}{\alpha} n, \tag{7}$$

for all integers $p \geqslant \alpha - 1$.

Proof. Let A be a 0, 1-matrix of size m by n with no submatrix of size α by β consisting entirely of ones. Let M be the total number of ones in A. The theorem will be proved if we can show

$$M \leqslant \frac{\beta - 1}{\binom{p}{\alpha - 1}}\binom{m}{\alpha} + \frac{(p + 1)(\alpha - 1)}{\alpha} n,$$

for all integers $p \geqslant \alpha - 1$.

Let j_i be the number of ones in the ith column of A. Then $M = \sum_1^n j_i$. If any column of A should contain less than $\alpha - 1$ ones we may arbitrarily add ones to that column until it has at least $\alpha - 1$ ones without the danger of creating a submatrix of size α by β consisting entirely of ones. Therefore, we may assume $j_i \geqslant \alpha - 1$ for all $i = 1,..., n$.

Now consider the set T of all m-tuples of zeros and ones each containing exactly α ones. We will say any $t \in T$ is incident with a column C of A if whenever there is a one in the vth place of t there is also a one in the vth place of C. Since A contains no submatrix of size α by β consisting entirely of ones we see that any $t \in T$ is incident with at most $\beta - 1$ columns of A. We now count incidences of elements of T and columns of A. The ith column of A is incident with exactly $\binom{j_i}{\alpha}$ elements of T, hence the total number of incidences is $\sum_1^n \binom{j_i}{\alpha}$. Therefore, we get

$$\sum_1^n \binom{j_i}{\alpha} \leqslant (\beta - 1)\binom{m}{\alpha}. \tag{8}$$

This yields, for real $a > 0$,

$$\sum_1^n a j_i(j_i - 1) \cdots (j_i - \alpha + 1) \leqslant a\alpha!\,(\beta - 1)\binom{m}{\alpha}$$

and since $M = \sum_1^n j_i$ we get

$$\sum_1^n [a j_i(j_i - 1) \cdots (j_i - \alpha + 1) - j_i] + M \leqslant a\alpha!\,(\beta - 1)\binom{m}{\alpha}$$

and for any real number c

$$\sum_1^n [a j_i(j_i - 1) \cdots (j_i - \alpha + 1) - j_i + c] + M \leqslant a\alpha!\,(\beta - 1)\binom{m}{\alpha} + cn. \tag{9}$$

We now wish to consider the polynomial

$$f(x) = ax(x - 1) \cdots (x - \alpha + 1) - x + c.$$

If we can show for appropriate choices of a and c that $f(x) \geqslant 0$ for all integers $x \geqslant \alpha - 1$ then (9) would yield

$$M \leqslant a\alpha!\,(\beta - 1)\binom{m}{\alpha} + cn. \tag{10}$$

To this end given any integer $p \geqslant \alpha - 1$ we choose a and c so that $f(p) = 0$ and $f(p + 1) = 0$. Once this is done we claim that $f(x) \geqslant 0$ for all integers $x \geqslant \alpha - 1$.

From $f(p) = f(p + 1) = 0$ we get

$$ap(p - 1) \cdots (p - \alpha + 1) - p + c = 0,$$
$$a(p + 1)p(p - 1) \cdots (p - \alpha + 2) - (p + 1) + c = 0,$$

so

$$a = \frac{1}{\alpha p(p - 1) \cdots (p - \alpha + 2)}; \qquad c = \frac{(\alpha - 1)(p + 1)}{\alpha}.$$

Note that $a > 0$ and

$$f'(x) = a\sum_{1}^{\alpha} x(x-1)\cdots\widehat{(x-s+1)}\cdots(x-\alpha+1) - 1$$

where the $\widehat{(x-s+1)}$ indicates that $x-s+1$ is deleted from the sth term in the sum. We see that $f'(x)$ is an increasing function for $x \geqslant \alpha - 1$. Therefore, $f(x)$ is a convex function for $x \geqslant \alpha - 1$ and since its only two roots are consecutive integers, the claim is established.

So we have shown that for $a = (1/\alpha p(p-1)\cdots(p-\alpha+2))$ and $c = ((p+1)(\alpha-1)/\alpha)$

$$M \leqslant a\alpha!\,(\beta-1)\binom{m}{\alpha} + cn$$

i.e.,

$$M \leqslant \frac{\beta-1}{\binom{p}{\alpha-1}}\binom{m}{\alpha} + \frac{(p+1)(\alpha-1)}{\alpha}n$$

Notice that Eq. (7) represents for fixed m, α, β and for varying p, an infinite family of inequalities each linear in n. Figure 1 shows the range of superiority for each p, as well as the relationship of Eq. (2) in case $\alpha = 2$.

Notice also that the right-hand side of (7) may not be an integer while, by definition, the left hand side must be. This implies that if (7) holds, so does

$$M_{\alpha,\beta}(m,n) \leqslant \left[\frac{\beta-1}{\binom{p}{\alpha-1}}\binom{m}{\alpha} + \frac{(p+1)(\alpha-1)}{\alpha}n\right], \qquad (7')$$

where $[x]$ is the greatest integer less than or equal to x. These same considerations apply to Theorems 3 and 4 as well.

3. Equality

By examining the proof of the theorem one sees that equality holds in (7) for a particular value of p whenever there is a matrix of size m by n with $\sum_1^n \binom{j_i}{\alpha} = (\beta-1)\binom{m}{\alpha}$ and with $j_i = p$ or $p+1$ for all $i = 1,\ldots,n$. That is whenever each member of T is incident with exactly $\beta - 1$ columns of the matrix and each column contains precisely p or $p+1$ ones.

By a tactical configuration $C[k, s, \lambda, v]$ we mean a system of subsets of a set E of cardinality v, having k elements each, such that every subset of E having s elements is contained in exactly λ of the sets of the system. We denote the number of sets in the system $C[k, s, \lambda, v]$ by $\bar{C}[k, s, \lambda, v]$. Then we see that equality holds in (7) for a particular p whenever the configuration $C[p, \alpha, \beta - 1, m]$ exists and $n = \bar{C}[p, \alpha, \beta - 1, m]$ or whenever the configuration $C[p + 1, \alpha, \beta - 1, m]$ exists and $n = \bar{C}[p + 1, \alpha, \beta - 1, m]$.

Wilson [9] has shown that given positive integers p and $\beta - 1$ a $C[p, 2, \beta - 1, m]$ exists for all sufficiently large m satisfying

$$\begin{aligned} (\beta - 1)(m - 1) &= 0 \bmod p - 1, \\ (\beta - 1)m\,(m - 1) &= 0 \bmod p(p - 1). \end{aligned} \tag{11}$$

This means that for any two positive integers p and $\beta - 1$ if m is sufficiently large and satisfies Eqs. (11) then

$$M_{2,\beta}(m, n) = \frac{\beta - 1}{p}\binom{m}{2} + \frac{(p + 1)}{2}n,$$

for $n = ((\beta - 1)/\binom{p}{2})\binom{m}{2}$. So equality holds in (7) with $\alpha = 2$ and β arbitrary for infinitely many values of m and n.

Furthermore, Hanani [5] has shown that, except for a $C[5, 2, 2, 15]$ which does not exist, if $3 \leqslant p \leqslant 5$ a necessary and sufficient condition for the existence of a $C[p, 2, \beta - 1, m]$ is that Eqs. (11) are satisfied. Hence equality holds in (7) with $\alpha = 2$, β arbitrary and $3 \leqslant p \leqslant 5$ whenever m satisfies (11) and $n = ((\beta - 1)/\binom{p}{2})\binom{m}{2}$.

Now if q is a prime power we know that the projective plane of order q exists. The incidence properties of such a plane are those of a tactical configuration with $m = n = q^2 + q + 1$. This implies that equality holds in (7) for $m = n = q^2 + q + 1$, $\alpha = \beta = 2$ and $p = q$ where q is any prime power. Similarly, the affine plane of order q, q a prime power, provides an example of equality in (7) for $m = q^2 + q, n = q^2, \alpha = \beta = 2$ and $p = q$. Also the inversive plane of order q, q a prime power, gives equality in (7) with $m = q^2 + 1, n = q(q^2 + 1), \alpha = 3, \beta = 2$ and $p = q$.

For general α we mentioned in section 1 that Čulík has shown equality holds in (7) for $p = \alpha - 1$ whenever $n \geqslant (\beta - 1)\binom{m}{\alpha}$.

We also have the result of R. K. Guy with a different proof:

THEOREM 2. *Let $T_{\alpha+1,m,\beta-1}$ be the maximum number of subsets of size $\alpha + 1$ that can be packed into a set of size m so that no subset of size α is in more than $\beta - 1$ of the subsets. Then for*

$$\max\left[\frac{\beta - 1}{\alpha + 1}\binom{m}{\alpha}, (\beta - 1)\binom{m}{\alpha} - T_{\alpha+1,m,\beta-1}\right] \leqslant n \leqslant (\beta - 1)\binom{m}{\alpha}$$

we have

$$M_{\alpha,\beta}(m,n) = \left[\frac{\beta-1}{\alpha}\binom{m}{\alpha} + \frac{\alpha^2-1}{\alpha}n\right]$$

That is, equality holds in (7′) *for* $p = \alpha$.

Proof. We define an appropriate matrix of size m by n as follows. Choose the first $t = [((\beta-1)/\alpha)\binom{m}{\alpha} - n/\alpha]$ columns to each contain exactly $\alpha + 1$ ones so that no m-tuple in T is incident with more than $\beta - 1$ columns. We may do this since

$$\frac{\beta-1}{\alpha+1}\binom{m}{\alpha} \leqslant n \quad \text{implies} \quad \frac{\beta-1}{\alpha}\binom{m}{\alpha} \leqslant \frac{\alpha+1}{\alpha}n,$$

so

$$t \leqslant \frac{\beta-1}{\alpha}\binom{m}{\alpha} - \frac{n}{\alpha} \leqslant n$$

and

$$(\beta-1)\binom{m}{\alpha} - \alpha T_{\alpha+1,m,\beta-1} \leqslant n$$

implies

$$(\beta-1)\binom{m}{\alpha} - n \leqslant \alpha T_{\alpha+1,m,\beta-1},$$

so

$$t \leqslant \frac{\beta-1}{\alpha}\binom{m}{\alpha} - \frac{n}{\alpha} \leqslant T_{\alpha+1,m,\beta-1}.$$

Now the number of m-tuples in T incident with these columns is

$$t\binom{\alpha+1}{\alpha} = \left[\frac{\beta-1}{\alpha}\binom{m}{\alpha} - \frac{n}{\alpha}\right](\alpha+1).$$

We may fill the remaining columns of the matrix each with exactly α ones in such a way as to avoid having any element of T incident with more than $\beta - 1$ columns since

$$\begin{aligned}(\beta-1)\binom{m}{\alpha} - \left[\frac{\beta-1}{\alpha}\binom{m}{\alpha} - \frac{n}{\alpha}\right](\alpha+1) &\geqslant \frac{\alpha+1}{\alpha}n - \frac{\beta-1}{\alpha}\binom{m}{\alpha}\\ &= n - t.\end{aligned}$$

TABLE 1[a]

m	n	$M_{2,2}(m, n)$	Right-hand side of (7′) for best choice of p.	m	n	$M_{2,2}(m, n)$	Right-hand side of (7′) for best choice of p.
8	8	24	25	12	19	60	60*
8	9	26	27	12	20	61	62
8	10	28	29	12	21	63	64
9	9	29	31	13	13	52	52*
9	10	31	33	13	14	53	54
9	11	33	34	13	15	55	56
				13	16	57	58
10	10	34	35	13	17	59	60
10	11	36	37				
10	12	39	39*	13	18	61	62
10	13	40	41	13	19	64	64*
10	14	42	43	13	20	66	66*
10	15	44	45	13	21	67	68
10	16	46	46*	13	22	69	70
10	17	47	47*				
				14	14	56	57
11	11	39	40	14	15	58	60
11	12	42	42*	14	16	60	62
11	13	44	44*	14	17	63	64
11	14	45	46	14	18	65	66
11	15	47	48				
				14	19	68	68*
11	16	50	50*	14	20	70	70*
11	17	51	52	14	21	72	72*
11	18	53	54	14	22	73	74
11	19	55	56	14	23	75	76
12	12	45	46	15	15	60	63
12	13	48	48*	15	16	63	66
12	14	49	50	15	17	66	68
12	15	51	52	15	18	69	71
12	16	53	54	15	19	72	73
12	17	55	56	15	20	75	75*
12	18	57	58	15	21	77	77*
				16	20	80	80*

[a] Places where Equality Holds in (7′) are Marked with an Asterisk (*). In some of these cases $p = 3$. The values of $M_{2,2}(m, n)$ are from R. K. Guy [4].

TABLE 2[a]

m	n	$M_{3,3}(m, n)$	Right-hand side of (7′) for best choice of p.
6	6	26	26*
6	7	29	30
6	8	32	33
6	9	36	36*
6	10	39	40
7	7	33	35
7	8	37	38
7	9	40	41
7	10	44	45
7	11	47	48
7	12	50	51
7	13	53	55
7	14	56	58
7	15	60	61
7	16	63	65
7	17	66	68
7	18	69	71
7	19	72	74
7	20	75	76
7	21	78	79
7	22	81	82
8	8	42	43
8	9	45	47
8	10	50	51
8	11	53	55
8	12	57	58
8	13	60	62
8	14	64	65
9	9	49	52
9	10	54	56
9	11	59	60
9	12	64	64*
10	10	60	64

[a] Places where equality holds in (7′) are marked with an asterisk (*). In some of these cases $p = 4$. The values of $M_{2,2}(m, n)$ are from R. K. Guy [4].

The total number of ones in this matrix is now

$$(\alpha+1)\left[\frac{\beta-1}{\alpha}\binom{m}{\alpha}-\frac{n}{\alpha}\right]+\alpha n-\alpha\left[\frac{\beta-1}{\alpha}\binom{m}{\alpha}-\frac{n}{\alpha}\right]$$
$$=\left[\frac{\beta-1}{\alpha}\binom{m}{\alpha}-\frac{n}{\alpha}\right]+\alpha n=\left[\frac{\beta-1}{\alpha}\binom{m}{\alpha}+\frac{\alpha^2-1}{\alpha}n\right] \quad \blacksquare$$

So we see equality holds in (7′) with $p=\alpha$.

Note that for $\alpha=2$ and $\beta=2$

$$T_{3,m,1}=\begin{cases}[\frac{1}{3}m[\frac{1}{2}(m-1)]] & \text{for} \quad m\neq 5 \bmod 6,\\ [\frac{1}{3}m[\frac{1}{2}(m-1)]]-1 & \text{for} \quad m=5 \bmod 6,\end{cases}$$

so $T_{3,m,1}$ is approximately equal to $\frac{1}{3}\binom{m}{2}$, and so equality holds in (7′) with $p=2$ roughly for

$$\frac{1}{3}\binom{m}{2}\leqslant n\leqslant\binom{m}{2},$$

which is the interval of supremacy for $p=2$ (see Fig. 1).

Tables 1 and 2 show that equality may hold for arbitrary small values of m and n even when $p=\alpha+1$ (here $\alpha=3,4$). But in these cases equality may also fail to hold.

4. Arrangements

An arrangement of n lines is defined to be any collection of n lines in the projective plane. For such arrangements we ask two questions.

First, given an arrangement of n lines and given any m distinct polygonal regions thereby determined, say $R_1,\ldots,R_m$, if we denote by $s(R_i)$ the number of edges on R_i then what is the maximum of $\sum_1^m s(R_i)$? We prove a generalization of a result of Canham [2].

Theorem 3. *If A is an arrangement of n lines, $R_1,\ldots,R_m$ m distinct polygonal regions determined by A and $p(R_i)$ the number of edges on R_i then*

$$\sum_1^m s(R_i)\leqslant\frac{4}{p}\binom{m}{2}+\frac{p+1}{2}n \qquad \textit{for all positive integers } p. \tag{13}$$

Proof. Consider the incidence matrix M of A defined as follows. M is a 0, 1-matrix of size m by n with a one in the i,jth place if and only if region R_i has an edge belonging to line L_j. Now it is an easy geometric fact that no 5 lines can each contribute an edge to 2 distinct regions. Therefore, M has no submatrix of size 2 by 5 consisting entirely of ones. So Eq. (7) holds with $\alpha = 2$ and $\beta = 5$. ∎

Canham noted that for $p = 1$ equality may be attained for each $n \geqslant 4\binom{m}{2}$. Little is known about the maximum value of $\sum_1^m s(R_i)$ for $n < 4\binom{m}{2}$. If we write $a(m, n) = \max \sum_1^m s(R_i)$, where the maximum is taken over all arrangements with n lines, then it is easy to check that $a(3, 6) = 15$, $a(3, 7) = 16$ and $a(3, 8) = 18$. The right hand side of (13) gives $a(3, 6) = 15$ and $a(3, 8) = 18$ with $p = 2$. When the greatest integer function is added, the right-hand side of (13) also gives $a(3, 7) = 16$. As far as the author knows, no other values of $a(m, n)$ are known for $n < 4\binom{m}{2}$.

Given an arrangement of n lines and given any m distinct vertices $v_1, \ldots, v_m$ of the arrangement we may also ask for the maximum of $\sum_1^m m(v_i)$, where $m(v_i)$ is the number of lines of the arrangement passing through v_i. We prove the following:

THEOREM 4. *If A is an arrangement of n lines, $v_1, \ldots, v_m m$ distinct vertices of A, with $m(v_i)$ equal to the number of lines of A passing through v_i, then*

$$\sum_1^m m(v_i) \leqslant \frac{1}{p}\binom{m}{2} + \frac{1}{2}(p + 1)n, \tag{14}$$

for all positive integers p.

Proof. Since no two lines of A can have the same two vertices on them the incidence matrix of lines and vertices defined analogously to that in the proof of Theorem 3 has no submatrix of size 2 by 2 consisting entirely of ones. Therefore, (7) holds with $\alpha = 2$ and $\beta = 2$. ∎

For $n \geqslant \binom{m}{2}$ the arrangements determined by m points, no 3 of which are collinear supply examples of equality in (14) for $p = 1$. If one takes the arrangement determined by m points in the plane, no 4 of which are collinear, then every column of the incidence matrix so obtained has $j_i = 2, 3$. Also, every m-tuple in T is incident with exactly one column of the matrix (i.e., every pair of the m points is on exactly one line) so equality holds in (14) for $p = 2$.

Burr, Grunbaum, Sloane [1] have provided examples of such arrangements with the number of collinear triples among the m points equal to $1 + [\frac{1}{6}m(m - 3)]$ hence with $n = \binom{m}{2} - 2(1 + [\frac{1}{6}m(m - 3)])$ which is approximately $((m + 3)/3(m - 1))\binom{m}{2} - 2$. Note that this is approximately the lower value ($n = \frac{1}{3}\binom{m}{2}$) of supremacy for $p = 2$. (see Fig. 1).

If one takes the examples of Burr, Grunbaum and Sloane and decreases the number of collinear triples by moving some points, examples of equality in (14) can be created with n larger than $\binom{m}{2} - 2(1 + [\frac{1}{6}m(m-3)])$. The exact values of n so obtained have not been established.

Finally, we note that since in the arrangement determined by any m points in the plane there must exist many (at least $(3/7)\, m$) lines containing only two of the m points, the proof of Theorem 1 indicates that it should be possible to improve Eq. (14).

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